POISSON'S AND LAPLACE'S EQUATION IN A GRAVITATIONAL FIELD →

Poisson's equation for gravitational field

Poisson's equation ⇒

If instead of the single point mass there are by a large number of discrete masses $m_1, m_2, m_3 --- m_i$ the force on mass "m" may be written as

$$\vec{F} = - \sum_i \frac{m m_i}{r_i^2} \hat{r}$$

where $\hat{r}$ is the unit vector in the direction along the line joining m and m_i

$$\vec{g} = \frac{\vec{F}}{m}$$

The gravitational field of the earth is caused by its density. The mass distribution of the planet is inherently three dimensional but we will always only graze at the surface. The most we can do is measure the gravitational acceleration at the Earth's surface.

However, thanks to a fundamental relationship known as Gauss's Theorem (Vector analysis), the link between a surface observable and the properties of the whole body (volume) in question can be found. It is also called the divergence theorem.

Actually Gauss's law is applicable to any situation that involves the inverse-square force law, since gravitational force is an using Gauss's divergence theorem.

$$\int_S \vec{g}\cdot\hat{n}\,da = \int_V \nabla\cdot\vec{g}\,dV$$

$$\int_V \nabla\cdot\vec{g}\,dV = -4\pi G\int_V \rho\,dV$$

$$\nabla\cdot\vec{g}\int_V dV = -4\pi G\rho\int_V dV$$

$$\nabla\cdot\vec{g} = -4\pi G\rho$$

$$\frac{\partial g_x}{\partial x} + \frac{\partial g_y}{\partial y} + \frac{\partial g_z}{\partial z} = -4\pi G\rho$$

$$\nabla\cdot(-\nabla V) = -4\pi G\rho$$

$$\nabla^2 V = 4\pi G\rho$$

$\therefore \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$\therefore \vec{g} = g_x i + g_y j + g_z k$

$\therefore \vec{g} = -\nabla V$

This is known as poisson's equation and is useful in a number of potential theory applications.

And in empty space $\rho = 0$

$$\boxed{\nabla^2 V = 0}$$

The result is an even better known equation called Laplace's equation.

Poisson's equation is useful in developing Green's function whereas we often encounter Laplace's equation when dealing with various co-ordinate systems.

Solid angle ($d\Omega$) is a two dimensional angle in 3D space. Solid angle ($d\Omega$) is representing the surface area element on the unit sphere. So, formally, solid angle is just the area subtended by the region on the unit sphere.

The integral $\int_S d\Omega$ represents a surface integral over the appropriate portion of the unit sphere.

At any point P on this surface, the total gravitational field is

$$\vec{g} = \vec{g}_1 + \vec{g}_2 + \vec{g}_3 + \cdots$$

while the total flux through the enclosed surface is

$$\phi_m = \int_S \vec{g} \cdot \hat{n}\, da = \int_S (\vec{g}_1 + \vec{g}_2 + \vec{g}_3 + \cdots)\, \hat{n}\, da$$

$$\int_S \vec{g} \cdot \hat{n}\, da = \int_S \vec{g}_1 \hat{n}\, da + \int_S \vec{g}_2 \hat{n}\, da + \int_S \vec{g}_3\, \hat{n}\, da + \cdots$$

inside the surface S by summing over the masses

$$\int_S \vec{g} \cdot \hat{n}\, da = -4\pi G \left[\sum_i m_i \right]$$

If we change to a continuous mass distribution within surface "S"

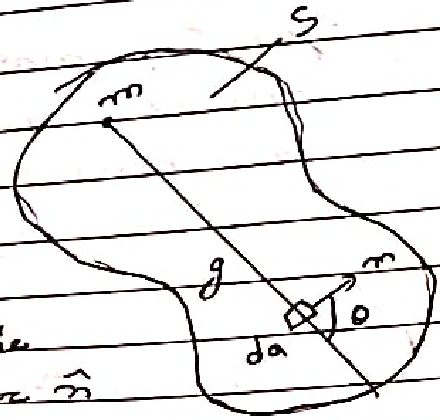
$$\int_S \vec{g} \cdot \hat{n}\, da = -4\pi G \int_V \rho\, dV$$

$\therefore \sum_i m_i = \int \rho\, dV$

where the integral on the right-hand side is over the volume "V" enclosed by "S", ρ is the mass density and dV is the differential volume.

inverse - square force. let apply Gauss's law and see its usefulness in calculating gravitational ~~force is an~~ field intensity $\vec{g}$ in simple situations

Consider an arbitrary surface with a mass "m" placed somewhere inside. let's find the gravitational flux ϕ emanating from mass "m" through the arbitrary surface "S"



$$\phi_m = \int_S \vec{g} \cdot d\vec{a}$$

$$\phi_m = \int_S \vec{g} \cdot \hat{n} \, da$$

where the integral is over the surface "S" and the unit vector $\hat{n}$ is normal to the surface at the differential area "da"

$$\phi_m = -Gm \int_S \hat{n} \cdot \frac{\hat{r}}{r^2} \, da \qquad \because \vec{g} = -G\frac{m}{r^2}\hat{r}$$

where $\hat{r}$ in the direction of $\vec{g}$

$$\phi_m = -Gm \int_S \frac{\cos\theta}{r^2} \, da \qquad \boxed{\because \hat{n}\hat{r} = (1)(1)\cos\theta}$$

where θ is angle between $\vec{g}$ and $\hat{n}$

The solid angle is the integral over of the arbitrary surface and has the value 4π steradians, which gives for the mass flux

$$\phi_m = \int_S \vec{g} \cdot \hat{n} \, da = -Gm(4\pi)$$

$\because$ Solid angle $= \int \frac{da\cos\theta}{r^2} = 4\pi$